Diversity and Technological Progress

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Abstract

This note proposes a tractable model to study the equilibrium diversity of technological progress and shows that equilibrium technological progress may exhibit too little diversity (too much conformity), in particular, foregoing socially beneficial investments in "alternative" technologies that will be used at some point in the future. The presence of future innovations that will replace current innovations imply that social benefits from innovation are not fully internalized. As a consequence, the market favors technologies that generate current gains relative to those that will bear fruit in the future; current innovations in research lines that will be profitable in the future are discouraged because current innovations are typically followed by further innovations before they can be profitably marketed. A social planner would choose a more diverse research portfolio and would induce a higher growth rate than the equilibrium allocation. The diversity of researchers is a partial (imperfect) remedy against the misallocation induced by the market. Researchers with different interests, competences or ideas may choose non-profit maximizing and thus more diverse research portfolios, indirectly contributing to economic growth.

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1 Introduction

Until the first decade of the 21st century, almost all research and product development in the transport industry were directed towards improving the power, design and fuel efficiency of vehicles using gasoline, even though it was widely recognized that those using alternative energy sources would have a large market in the future as oil prices increased and consumers became more environmentally conscious.[1] Investment in a variety of other alternative energy sources were similarly delayed.[2] Although many commentators now decry the delays in the development of viable alternatives to fossil fuels, it is difficult to know what the marginal rate of private and social returns to different types of research were in these sectors, and thus whether the amount of *diversity* generated by the market economy was optimal.

As a first step in the investigation of these issues, this note theoretically investigates whether the market economy provides adequate incentives for research in alternative technologies—as opposed to technologies that are currently and extensively used. Put differently, I ask whether the market economy will achieve the efficient amount of *diversity* in research or whether it will tend to encourage research to be excessively concentrated in some research lines and products.

The main contributions of this note are twofold. The first is to develop a dynamic model of innovation that can be used to analyze the issues of equilibrium and optimal amounts of diversity of technological progress. The second is to use this model to show that there is a natural mechanism leading to "too little diversity". I also suggest that a counteracting force against the potential lack of diversity in research may be the "diversity of researchers": because of different competences, beliefs or preferences, researchers may choose to direct their research towards areas that are under-explored by others and this may partially redress the inefficiently low level of diversity of research in the market economy.

The mechanism at the heart of this note is as follows: given the patent system we have in place, an innovation creates *positive externalities* on future innovations that will build on its discoveries and advances. The patent system makes sure that no other firm can copy the current innovation and requires an innovation to be different from "prior art" in the area (see, for example, Scotchmer, 2005). However, provided that a certain "required inventive step" is exceeded, a new innovation, even if it builds on prior patented knowledge, would not have to make royalty payments. In fact, an important objective and a great virtue of the patent system is to make knowledge freely available to future innovators and thus some amount of "building on the shoulders of past innovations" is clearly both desirable and unavoidable. In addition,

[1] Crosby (2006) and Roberts (2005) for readable accounts of the history of research on different of energy sources.

[2] For example, as of 2006, more than 80% of all world energy consumption is from fossil fuels and less than 1% from geothermal, wind and solar combined (International Energy Agency, 2008).

patent life is capped at 20 years, so even externalities created on further innovations that do not meet the inventive step requirement cannot be fully internalized. The key observation here is that this positive externality on future innovations will affect different types of innovations differentially.

Consider two potential products, a and b, which are competing in the market. Suppose that product a has higher quality, so that all else equal, consumers will buy product a. However, at some future date, consumer tastes (or technology) will change so that product b will become more popular. We can think of product a as vehicles using fossil fuels and product b as "electric cars" or other "clean technology vehicles". Consider two types of innovations. The first, innovation A, will lead to a higher-quality version of product a and thus the output of this innovation can be marketed immediately. Even though it creates positive externalities on future products that can build on the processes that it embeds, innovation A still generates a profit stream and this will typically encourage some amount of research. Contrast this to innovation B, which leads to a higher quality of product b and thus can only be marketed after tastes change. Improvements in product b will be useful for the society in the future (because tastes will indeed change at some point). But private incentives for innovation B are weak because the innovator is unlikely to benefit from the improvements in the quality of product b even in the future because some other innovation is likely to significantly improve over the current one before tastes change.

The scenario described above highlights a general feature: the recognition that there will be further innovations will discourage research in areas that will generate new products or technologies for the future relative to improving currently used products, processes, or technologies. Consequently, in equilibrium, too much research will be devoted to currently successful product and technology lines—in the above example, innovation A. I refer to this situation as "*lack of diversity in research*" (or alternatively as "too much conformity").

This note shows how these ideas can be formalized using a dynamic model of innovation. Using this model, it formalizes the ideas discussed above and clarifies the conditions under which there will be too little diversity in research. In particular, it shows that provided that the probability (flow rate in continuous time) of changes in tastes is sufficiently high, the market equilibrium involves too little diversity and too little growth.

As the discussion here illustrates, this pattern is predicated on a specific patent system. Naturally, an alternative patent system that internalizes all positive externalities created on future innovations would solve this problem. However, such a patent system is different from what we observe in practice and also difficult to implement. For example, such a patent system would require all innovations in laser technology or solid-state physics to make royalty payments to Heisenberg, Einstein and Bohr or all steam engine innovations to make payments to Newcomen and Watts (or to their offspring).

While the analysis here suggests that in an idealized economy there will be too little—in fact no—diversity in research even though innovations being directed at a wider set of research lines is socially optimal, in practice a society may generate a more diverse set of research output because of "*diversity of researchers*". In particular, if the society has or generates a set of researchers with different competencies, preferences and beliefs, then part of its research effort will be directed at alternative products and technologies rather than all effort being concentrated on current technology leaders. For instance, in the context of the above example, even though incentives to improve product a may be greater than those for product b, some researchers may have a comparative advantage in the type of research that product b requires or may have heterogeneous beliefs, making them more optimistic about the prospect of a change in tastes, thus strengthening their desire to undertake research for product b. Although this kind of researcher diversity will not restore the Pareto optimal amount of diversity in research, it will act as a countervailing force against the market incentives that imply too much homogenization. Thus the analysis here also suggests why having a more diverse set of researchers and scientists might be useful for a society's long-run technological progress and growth potential. This intuition is formalized by showing that a greater diversity in the competences of researchers increases research directed at substitute varieties and the equilibrium rate of economic growth.

Popular discussions often emphasize the importance of diversity in various settings, including in research, and also stress that non-profit motives are important in research. The framework here offers a simple formalization of both ideas: diversity in research is important for economic growth but the market economy may not provide sufficient incentives for such diversity; diversity of researchers, in fact their non-profit-seeking or "nerdy" behavior, may be socially beneficial as a remedy for the lack of diversity (too much conformity) in research.

The model used here is related to the endogenous technological change literature, for example, Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992). This literature typically does not investigate the diversity of research. In addition, the "lock-in" effects in technology choices emphasized by Arthur (1989), and the subsequent literature building on his work, are closely related to the main mechanism leading to too little equilibrium diversity in the current model, though the modeling approaches are very different (the approach here builds on endogenous technological change models with forward-looking innovation decisions, while Arthur's framework relies on learning by doing externalities; see also Katz and Shapiro, 1986).

A smaller literature investigates the determinants of research and innovative activity. Aghion and Tirole (1994) is an early contribution, focusing on incentive problems that arise in the management of innovation. More recently, Brock and Durlauf (1999) investigate equilibrium choice of research topic in scientific communities. Aghion, Dewatripont and Stein (2007) analyze the implications of academic freedom, while Murray, Aghion, Dewatripont, Kolev and Scott (2008) empirically investigate the effect of scientific openness on further research. Jones (2009) argues that scientific research is becoming more difficult because there is now larger body of existing knowledge that needs to be absorbed and shows how this can explain why major breakthroughs happen later in the lives of scientists and why scientific collaborations have become more common. Bramoullé and Saint-Paul (2008) construct a model of research cycles, where equilibrium research fluctuates between invention of new lines of research and development of existing lines. Acemoglu, Bimpikis and Ozdaglar (2008) propose a model where firms might have incentives to delay research and copy previous successful projects rather than engage in simultaneous search. None of these papers highlight or analyze the issues related to diversity in research emphasized in this note. The only other works that discuss issues of diversity in research are Bronfenbrenner (1966), Stephan and Levin (1992), and Sunstein (2001), who emphasize the possibility of fads in academic research.

The rest of this note is organized as follows. Section 2 provides a simple example illustrating the basic idea. Section 3 presents the baseline environment and characterizes the equilibrium. Section 4 characterizes the conditions under which the equilibrium will be inefficient and technological progress will be too slow because of inefficiently low levels of diversity in research. Section 5 characterizes the equilibrium when there is diversity in research tastes and shows how greater diversity increases economic growth. Section 6 concludes, while the Appendix provides an extended environment that motivates some of the simplifying assumptions in the main text. It also contains some additional derivations.

2 A Simple Example

In this section, I provide a simple example illustrating the main mechanism leading to inefficiently low diversity in research. Consider a two-period economy, with periods $t = 1$ and $t = 2$ and no discounting. There are two technologies j and j', both starting $t = 1$ with qualities $q_j(0) = q_{j'}(0) = 1$. A scientist can work to improve both technologies. Suppose that the scientist has a total of one unit of time. The probability of improving either of the two technologies when he devotes x units of his time to the technology is $h(x)$. Suppose that h is strictly increasing, differentiable, and concave, and satisfies the Inada condition that $\lim_{x\to 0} h'(x) = \infty$. An improvement increases the quality of the technology (j or j') to $1+\lambda$ (with $\lambda > 0$). At $t = 1$, j is the "active" technology, so if the scientist improves technology j, he will be able to market it and receive return equal to $1+\lambda$. Technology j' is not active, so even if the scientist improves this technology, he will not be able to market it at $t = 1$. At time $t = 2$, technology j' becomes active with probability $p > 0$, replacing technology j (and if so, technology j can no longer be marketed). Before either of the two technologies is marketed at $t = 2$, other scientists can further improve over these technologies. Suppose that this happens with (exogenous) probability $v \in (0, 1]$ (and assume, for simplicity, that no such further improvements are possible if there is no innovation by the scientist at $t = 1$). In this event, the quality of the product increases by another factor $1 + \lambda$, but the original scientist receives no returns.[3]

Let us consider the problem of the scientist in choosing the optimal allocation of his time between the two research projects. When he chooses to devote $x_j \in [0, 1]$ units of his time to technology j, his return can be written as

$$\begin{aligned} \pi(x_j) &= h(x_j)\left[1 + (1-p)(1-v)\right](1+\lambda) \\ &\quad + h(1-x_j)\, p\,(1-v)(1+\lambda). \end{aligned} \tag{1}$$

The first line is the scientist's expected return from innovation in technology j. He is successful with probability $h(x_j)$ and receives immediate returns $1 + \lambda$. In the next period, technology j remains active with probability $1 - p$ and his innovation is not improved upon with probability $1 - v$, and in this event, he receives $1 + \lambda$ again. With probability $h(1 - x_j)$, he successfully undertakes an innovation for technology j'. Since this technology is not yet active, he receives no returns at $t = 1$, but if it becomes active at $t = 2$ (probability p) and is not improved upon (probability $1 - v$), he will receive $1 + \lambda$ at $t = 2$.

Maximizing $\pi(x_j)$ with respect to x_j gives the following simple first-order condition:

$$h'\left(x_j^*\right)\left[1 + (1-p)(1-v)\right] = h'\left(1 - x_j^*\right) p\,(1-v). \tag{2}$$

Clearly, x_j^* is uniquely defined. It can also be verified that it is increasing in v; as the probability of further innovations increases, more of the scientist's time will be devoted to technology j. Also notably as $v \to 1$, $x_j^* \to 1$ and all research is directed to the currently active technology, j. The intuition for this result is simple. Because of future improvements, as $v \to 1$, the scientist will receive no returns from innovation in technology j'—somebody else will have invented even a better version of this technology by the time it can be marketed. There will be a similar improvement over technology j if it remains active until $t = 2$, but the scientist will in the meantime receive returns from being able to market it immediately (during $t = 1$). Thus the prospect of future improvements over the current innovation disproportionately favors the currently-active technology. This intuition also explains why x_j^* is increasing in v.

[3] Thus there are no patents that make further innovations pay royalties to the original inventor. Patent systems and how they affect the results are discussed in the next section.

For comparison, let us consider the research allocation choice of a planner wishing to maximize total value of output. This can be written as

$$\begin{aligned}\Pi\left(x_j\right) &= h\left(x_j\right)\left[\left(1+\left(1-p\right)\left(1-v\right)\left(1+\lambda\right)\right)+\left(1-p\right)v\left(1+\lambda\right)^2\right] \\ &\quad +h\left(1-x_j\right)\left[p\left(1-v\right)\left(1+\lambda\right)+pv\left(1+\lambda\right)^2\right].\end{aligned}$$

This differs from (1) because the planner also benefits when there is another innovation building on the shoulders of the innovation of the scientist at $t = 1$. The allocation of time between the two technologies that would maximize $\Pi\left(x_j\right)$ is given by the following first-order condition:

$$h'\left(x_j^S\right)\left[\left(1+\left(1-p\right)\left(1-v\right)\right)+\left(1-p\right)v\left(1+\lambda\right)\right]=h'\left(1-x_j^S\right)\left[p\left(1-v\right)+pv\left(1+\lambda\right)\right].$$

It can be verified that $x_j^S < x_j^*$, so that the social planner would always prefer to allocate more of the scientist's time to technology j' (and thus less of his time to the active technology). Interestingly, x_j^S is decreasing in v. In particular, even as $v \to 1$, $x_j^S > 0$. Intuitively, the social planner values the improvements in technology j' more than the scientist because the society will benefit from further improvements over those undertaken by the scientist at time $t = 1$. In fact, as v increases, future improvements become *more important* to the social planner relative to current gains, favoring research directed at technology j'. The scientist does not value such improvements because they deprive him of the returns from his innovation. Consequently, the choice by the scientist—relative to the allocation desired by the social planner—leads to too little "diversity" in the sense that the majority (or when $v \to 1$, all) of his research effort is devoted to the active technology.

Finally, it is straightforward to extend this environment by including several scientists. When all scientists have the same preferences and maximize their returns, the results are similar to those discussed here. However, when some scientists have different preferences and prefer to work on technology j', or have different beliefs and are more optimistic about a switch from technology j to technology j', then this type of "diversity of researchers"—and the associated "non-profit maximizing" behavior—will redress some of the inefficiency due to too little "diversity in research".

The next section provides a more detailed model that develops these intuitions.

3 Model

In this section, I introduce the baseline environment and characterize the equilibrium. The baseline environment is chosen to highlight the main economic mechanism in the most transparent manner. Subsection 3.4 discusses why the specific modeling assumptions were chosen.

The Appendix shows how similar results can be derived in a richer environment building on endogenous technological change models.

3.1 Description of Environment

Time is continuous and indexed by $t \in [0, \infty)$. Output is produced as an aggregate of a continuum of intermediate goods (products), with measure normalized to 1. Each intermediate $\nu \in [0, 1]$ comes in several (countably infinite number of) varieties, denoted by $j_1(\nu)$, $j_2(\nu)$,.... Variety $j_i(\nu)$ of intermediate ν has an endogenous quality $q_{j_i}(\nu, t) > 0$ (at time t). The quality of each variety is determined by the position of this product in a *quality ladder*, which has rungs equi-proportionately apart by an amount $1 + \lambda$ (where $\lambda > 0$). Thus for each $j_i(\nu)$, we have

$$q_{j_i}(\nu, t) = (1 + \lambda)^{n_{j_i}(\nu,t)} q_{j_i}(\nu, 0),$$

with $n_{j_i}(\nu, t) \in \mathbb{Z}_+$ corresponding to the rung of this product on the quality ladder. Throughout, let us normalize $q_{j_i}(\nu, 0) = 1$ for all $\nu \in [0, 1]$ and $i = 1, 2, ...$ Product qualities increase due to technological progress driven by research, which raises the rung of the product in the quality ladder. I describe the process of technological progress below.

At any point in time, only one of the varieties of any intermediate $\nu \in [0, 1]$ can be used in production. I use the notation j (or $j(\nu)$) to denote this "*active*" variety. Aggregate output is therefore given by

$$Y(t) = Q(t) \equiv \int_0^1 q_j(\nu, t)\, d\nu, \tag{3}$$

where $Q(t)$ is the average quality of active intermediates at time t. The production function (3) is a reduced-form representation of several richer endogenous growth models.[4]

Because of switches in tastes or other technological changes, the active variety of each intermediate becomes obsolete ("disappears") at the flow rate $\alpha \geq 0$ at any point. These obsolescence events are independent across intermediates and over time. The motivation for this type of obsolescence is the switch in technology induced by environmental concerns from the "active" technologies based on fossil fuels to "substitute" alternative energy sources discussed in the Introduction. In particular, let us order varieties such that if $j_i(\nu)$ is the active variety of intermediate ν at t, then when it disappears the active variety becomes $j_{i+1}(\nu)$. With a slight abuse of notation, at any point in time I use j to denote the currently active variety and j' to denote the next variety.

There is a continuum of "scientists," with measure normalized to 1. A scientist can work either on active varieties or on substitute varieties.[5] A scientist working on active varieties

[4] The Appendix sketches one such model, which leads to a structure identical to the reduced-form model used here.

[5] See the Appendix for a model in which research is also directed to specific intermediates.

discovers a higher quality version of one of the intermediates at the flow rate $\eta > 0$. Which intermediate the innovation will be for is determined randomly with uniform probability. The quality ladder structure introduced above implies that an innovation starting from an intermediate of quality q leads to a new quality equal to $(1+\lambda)\,q$.

A scientist working on substitute varieties discovers a higher quality version of the next-in-line substitute for one of the intermediates at the flow rate $\zeta\eta$, where $\zeta \geq 1$ (again chosen with uniform probability). This assumption implies that if the current active variety is $j_i(\nu)$ for intermediate ν, then substitute research could lead to the invention of a higher quality version of $j_{i+1}(\nu)$. Following such a discovery, the quality of the substitute variety increases from q' to $(1+\lambda)\,q'$. The presence of the term ζ allows innovation for substitute varieties to be easier than innovation for active varieties, for example, because the availability of a more advanced active variety makes some of these improvements for the related substitute variety more straightforward to discover or implement. Since improvements in the quality of substitute varieties also take the form of moving up in the rungs of the quality ladder, we can summarize the quality differences between active and substitute varieties by the difference in the number of steps ("quality gap") in the ladder between the two, which I will denote by $n(\nu,t)$ or simply by $n(\nu)$ or n. Formally,

$$n(\nu,t) \equiv n_{j_i}(\nu,t) - n_{j_{i+1}}(\nu,t)\,.$$

In addition to endogenous quality improvements, there are "exogenous" quality improvements for all substitute varieties. In particular, if variety $j_i(\nu)$ is the active one for intermediate ν, then I assume that any $i+1 > i$ cannot be more than N steps behind the currently active variety $j_i(\nu)$. In other words, $q_{j_{i+1}}(\nu,t)$ cannot be less than $\gamma q_{j_i}(\nu,t)$ when the quality of the active variety is $q_{j_i}(\nu,t)$, where $\gamma \equiv (1+\lambda)^{-N}$ for some $N \in \mathbb{N}$ (and thus $\gamma < 1$). The specification in particular implies that if the quality of the active variety increases from $q_{j_i}(\nu,t)$ to $q_{j_i}(\nu,t+) = (1+\lambda)\,q_{j_i}(\nu,t)$ and we have $q_{j_{i+1}}(\nu,t) = \gamma q_{j_i}(\nu,t)$, then the quality of the substitute variety $i+1$ also increases to $q_{j_{i+1}}(\nu,t+) = \gamma(1+\lambda)\,q_{j_i}(\nu,t)$.[6] As a consequence, when a switch (from j to j') happens we always have $q_{j'}(\nu,t) \geq \gamma q_j(\nu,t)$. Furthermore, suppose throughout that $q_{j'}(\nu,t) \leq q_j(\nu,t)$ for all $\nu \in [0,1]$—substitute varieties cannot be more advanced than active varieties.

What about the gap between the substitute variety $j_{i+1}(\nu)$ and its substitute $j_{i+2}(\nu)$? I assume that research on substitute varieties creates a positive spillover on the quality of varieties beyond the immediate substitute (in particular, on $i+2$), so that when be the gap between $i+1$ and i is n so will the gap between $i+2$ and $i+1$, i.e., $n_{j_{i+1}}(\nu,t) - n_{j_{i+2}}(\nu,t) = n_{j_i}(\nu,t) - n_{j_{i+1}}(\nu,t)$. This assumption simplifies the analysis by allowing for an explicit characterization

[6] The notation $q_{j_i}(\nu,t+)$ stands for $q_{j_i}(\nu,t)$ just after time t.

of the stationary distribution of quality gaps.[7]

I also assume that

$$\zeta \leq \gamma^{-1}, \tag{4}$$

so that the relative ease of innovation in substitute varieties does not exceed the productivity advantage of the active varieties.

The patent system functions as follows. A scientist who has invented a higher quality (of the active variety) of some intermediate has a perfectly enforced patent and receives a revenue equal to the contribution of its intermediate to total output. That is, a scientist with a patent on the active variety of an intermediate with quality $(1+\lambda)\,q$ receives a flow revenue of λq, since the contribution of this intermediate to total output over the next highest quality, q, is $(1+\lambda)\,q - q = \lambda q$.[8] Importantly, an improvement over this variety (for example, leading to quality $(1+\lambda)\,q$) does not constitute a patent infringement and thus the scientist in question does not receive any revenues after another scientist improves the quality of this variety. Similarly, scientists that undertake inventions improving the quality of the substitute variety are also awarded a *perfectly enforced patent* and can receive a flow revenue of q for their product of quality q if (and after) the active variety of this intermediate disappears. Also, suppose that if there is a further innovation for the active variety, from $q_j(\nu,t)$ to $q_j(\nu,t+) = (1+\lambda)\,q_j(\nu,t)$, then the next-in-line substitute variety of intermediate ν of quality $q_{j'}(\nu,t+) = (1+\lambda)\,\gamma q_j(\nu,t+)$ becomes freely available. Consequently, subsequent to such an innovation in the active variety, all substitute varieties with quality $q \leq q_{j'}(\nu,t+)$ would receive no revenues even if the active variety were to disappear. Therefore, only holders of a patent for substitute varieties of quality $q_{j'}(\nu,t) > \gamma q_j(\nu,t)$ will receive revenues when the active variety disappears.[9]

Finally, let us assume that scientists maximize the (expected) net present discounted value of their revenues with discount rate $r > 0$.

Given the above description, an *equilibrium* in this economy is given by a time path of research decisions by scientists that maximize their net present discounted values (in particular, they choose whether to undertake research directed at active or substitute varieties) and the

[7]In fact, all that is necessary is the weaker assumption that $n_{j_{i+1}}(\nu,t) - n_{j_{i+2}}(\nu,t)$ has the same stochastic distribution as $n_{j_i}(\nu,t) - n_{j_{i+1}}(\nu,t)$. This will be the case, for example, under the following scenario: using the same notation as below, let p_u denote the flow rate of innovation in the active variety and p_d denote the flow rate of innovation in the next in line substitute variety; then the flow rate of innovation in the substitute of the substitute needs to be approximately p_d^2/p_u (see equation (12)).

[8]This expression assumes that the current scientist is not the holder of the next highest quality. As shown below, this is without loss of any generality.

More generally, we could assume that the scientist receives a flow revenue of $\beta\lambda q$ for some $\beta \in (0,1]$, with identical results. The model presented in the Appendix corresponds to the case in which $\beta \in (0,1)$.

[9]Here I am using the fact that $\gamma \equiv (1+\lambda)^{-N}$. Without this feature, improvements in the quality of the active variety of some intermediate may reduce the potential contribution of the substitute varieties that have quality $q_{j'}(\nu,t) \in (\gamma q_j(\nu,t), (1+\lambda)\,\gamma q_j(\nu,t))$. When $\gamma \equiv (1+\lambda)^{-N}$, $q_{j'}(\nu,t) > \gamma q_j(\nu,t)$ we automatically have $q_{j'}(\nu,t) \geq \gamma(1+\lambda)\,q_j(\nu,t)$.

distribution of technology gaps between sectors. More formally, let $\omega(t) \in [0,1]$ be the fraction of researchers at time t undertaking research in substitute varieties and $\mu_n(t) \in [0,1]$ be the fraction of intermediates where the gap between the active variety $j_i(\nu)$ and the next substitute variety $j_{i+1}(\nu)$ is $n = 0, 1, ..., N$ steps. An equilibrium can then be represented by time paths of $\omega(t)$ and $\mu_n(t)$ (for $n = 0, 1, ..., N$). A *stationary equilibrium* is an allocation in which $\omega(t) = \omega^*$ and $\mu_n(t) = \mu_n^*$ (for $n = 0, 1, ..., N$) for all t. I focus on stationary equilibria.

3.2 Equilibrium

In this subsection, I show that all scientists undertaking research on active varieties, that is, $\omega(t) = 0$ for all t, is a stationary equilibrium. I then provide conditions for this to be a unique equilibrium.

Consider such a candidate (stationary) equilibrium. Then the value of holding the patent to the active intermediate of quality $(1+\lambda)q$ is

$$rV(q) = \lambda q - (\alpha + \eta) V(q). \tag{5}$$

This is intuitive. The scientist receives a revenue of λq $(= (1+\lambda)q - q)$ until the first of two events: (i) there is a switch to the substitute technology, which takes place at the flow rate α, or (ii) there is a new innovation, which happens at the flow rate η (since all scientists work to improve active varieties, the total measure of scientists is 1, and the measure of intermediates is normalized to 1).[10] Following both events, the scientist loses his patent on this product. Therefore, the right-hand side of (5) must be equal to the discount rate, r, times the value of the patent.[11] Note also that given the large number ("continuum") of other scientists, the likelihood that he will be the one inventing the next highest quality is zero. This also explains why assuming that patents on the highest and the next highest qualities are never held by the same scientist is without loss of any generality.[12] Equation (5) gives the value of holding the patent for active intermediate of quality q as:

$$V(q) = \frac{\lambda q}{r + \alpha + \eta}. \tag{6}$$

[10] Note that $V(q)$ refers to the value of the patent, not to the continuation value of the scientist in question; a scientist is undertaking parallel research regardless of whether this product is replaced or not. Following the disappearance of the active variety or another innovation, the value of the patent disappears, explaining the last term in (5).

[11] More generally, $\dot{V}(q)$ should be subtracted from the left-hand side, but under this candidate stationary equilibrium, we have $\dot{V}(q) = 0$.

[12] If, as in the model in the Appendix, we were to allow research to be directed to specific intermediates, then a standard argument based on Arrow's replacement effect (Arrow, 1962) would immediately imply that scientists never wish to undertake research on the intermediate line in which they have the best product. See Acemoglu (2009) for a textbook treatment of Arrow's replacement effect.

Therefore, the value of directing research to active varieties can be written as

$$R^A(Q) = \eta \int_0^1 V(q(\nu,t))\, d\nu = \frac{\eta\lambda Q}{r+\alpha+\eta}, \tag{7}$$

where recall that $Q \equiv \int_0^1 q_j(\nu)\, d\nu$. In particular, such research will lead to a successful innovation at the flow rate η for one of the intermediates (chosen uniformly). When previously this intermediate had quality q, the innovation will produce a version of the same intermediate with quality $(1+\lambda)q$ and will yield value $V(q)$, as given by (6), to scientists.

Before characterizing the value of directing research towards substitute varieties, we need to determine the distribution of intermediates by technology gap between active and next-in-line substitute varieties (the μ_n's). Since we are focusing on stationary equilibria, the fraction of researchers working towards innovations in the substitute varieties is constant (ω). Given ω, I now characterize the stationary distribution of μ_n's.[13] Define $p_u \equiv \eta(1-\omega)$ and $p_d \equiv \zeta\eta\omega$ as the flow rates of innovation of the active and substitute varieties, respectively. Then

$$(p_u + p_d)\,\mu_n = p_u\mu_{n-1} + p_d\mu_{n+1} \qquad \text{for } \; n = 1, ..., N-1. \tag{8}$$

Intuitively, total exits from state n (for $n = 1, 2, ..., N-1$) have three sources. First, there may be an innovation among the active varieties of the intermediates with gaps of n steps, which takes place at the flow rate p_u. Secondly, there may be an innovation among the substitute varieties of intermediates with gaps of n steps, which takes place at the flow rate p_n. Thirdly, one of the active varieties with gaps of n steps may disappear, which takes place at the flow rate α. This makes total exits from state n equal to

$$(p_u + p_d + \alpha)\,\mu_n.$$

With a similar reasoning, entry into this state comes from three sources. Either there has been an innovation in the active variety in intermediates with gaps of $n-1$ (flow rate p_u times μ_{n-1}); or there has been an innovation in the substitute varieties of intermediates with gaps of $n+1$ (flow rate p_d times μ_{n+1}); or an active variety (of any gap) has disappeared. In this last case, if the active variety $j_i(\nu)$ of intermediate ν has disappeared and been replaced by $j_{i+1}(\nu)$, then the relevant gap becomes the same as that between $j_{i+1}(\nu)$ and $j_{i+2}(\nu)$, but by assumption, this is the same as the gap between $j_i(\nu)$ and $j_{i+1}(\nu)$, so this last source of entry contributes $\alpha\mu_2$, to give us total entry into state n as

$$p_u\mu_{n-1} + p_d\mu_{n+1} + \alpha\mu_n.$$

Combining this with the previous expression gives (8).

[13] The Appendix characterizes the evolution of the distribution of quality gaps when $\omega(t)$ is time varying.

Equation (8) does not apply at the boundaries, since the gap cannot fall below 0 and cannot increase above N. In these cases, with a similar reasoning, we have

$$p_u \mu_0 = p_d \mu_1, \tag{9}$$

and

$$p_u \mu_{N-1} = p_d \mu_N. \tag{10}$$

In addition, by definition

$$\sum_{n=0}^{N} \mu_n = 1. \tag{11}$$

Equations (8)-(11) define the stationary distribution of a continuous-time Markov chain. Since this Markov chain is aperiodic and irreducible, it has a unique stationary distribution (e.g., Norris, 1998), which can be directly computed as

$$\mu_n^* = \left(\frac{p_d}{p_u}\right)^{N-n} \left(\sum_{j=0}^{N} \left(\frac{p_d}{p_u}\right)^{N-j}\right)^{-1} \qquad \text{for } n = 0, ..., N,$$

or written as a function of ω, as

$$\mu_n^* (\omega) = \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n} \left(\sum_{j=0}^{N} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-j}\right)^{-1} \qquad \text{for } n = 0, ..., N. \tag{12}$$

Under the candidate equilibrium studied here, there is no other research directed to substitute varieties and thus $\omega = 0$. Then (12) gives the stationary distribution of quality gaps as

$$\mu_N^* = 1 \text{ and } \mu_n^* = 0 \text{ for } n = 0, 1, ..., N-1.$$

Therefore, the gap between the active and substitute varieties of any intermediate will be N steps. Given this stationary distribution, we can next characterize the return to undertaking research directed at substitute varieties. Suppose, in particular, that a scientist directs his research to the substitute varieties. Under the candidate equilibrium, the quality of the substitute variety for intermediate ν is

$$q'(\nu, t) = \gamma q(\nu, t),$$

when the quality of the active product is $q(\nu, t)$. A successful innovation on a substitute variety of quality q' leads to a product of quality $(1+\lambda) q'(\nu, t) = (1+\lambda)\gamma q(\nu, t)$, but this is still a substitute variety and will remain so until the active variety disappears. A patent on this product therefore has value $\tilde{V}(q')$ such that

$$r\tilde{V}(q') = \alpha \left(V(q') - \tilde{V}(q')\right) - \eta \tilde{V}(q'). \tag{13}$$

Intuitively, this patent does not provide any revenues until the active variety disappears, which takes place at the flow rate α. However, if there is an additional innovation on the active variety of this intermediate before this event, the active variety increases its quality to $(1+\lambda)\,q$ and the substitute variety of quality $(1+\lambda)\,\gamma q = (1+\lambda)\,q'$ becomes freely available and thus the patent on this variety is no longer valuable (see also the explanation for equation (17) in the next subsection). Therefore, we have

$$\tilde{V}\left(q'\right) = \frac{\alpha V\left(q'\right)}{r+\alpha+\eta} \tag{14}$$

and the return to undertaking research on the substitute varieties when the average quality of active varieties is $Q \equiv \int_0^1 q_j\left(\nu\right) d\nu$ can be written as

$$\begin{aligned} R^S\left(Q\right) &= \frac{\zeta\eta\alpha}{r+\alpha+\eta}\int_0^1 V\left(\gamma q\left(\nu,t\right)\right) d\nu \\ &= \frac{\zeta\eta\alpha}{r+\alpha+\eta} \times \frac{\lambda\gamma Q}{r+\alpha+\eta} \\ &= \frac{\zeta\alpha\gamma}{r+\alpha+\eta} R^A\left(Q\right). \end{aligned} \tag{15}$$

Since $\zeta \leq \gamma^{-1}$ (from (4)), $r > 0$, $\eta > 0$, and $\alpha > 0$, comparison of (7) and (15) immediately establishes that $R^A\left(Q\right) > R^S\left(Q\right)$, so that the candidate equilibrium is indeed an equilibrium and no scientist undertakes research on substitute varieties. Intuitively, the fact that substitute varieties only become marketable at some future date (stochastically arriving at the flow rate α) makes research directed at them relatively unattractive compared to research on active varieties, which, when successful, will have immediate returns.[14]

Let us next compute the equilibrium growth rate. First note that for any $\nu \in [0,1]$ and for Δt sufficiently small, quality $q\left(\nu, t+\Delta t\right)$ will increase to $(1+\lambda)\,q\left(\nu,t\right)$ with probability $\eta\Delta t + o\left(\Delta t\right)$, it will fall to $\gamma q\left(\nu,t\right)$ with probability $\alpha\Delta t + o\left(\Delta t\right)$, and will remain constant at $q\left(\nu,t\right)$ with probability $1-\eta\Delta t - \alpha\Delta t + o\left(\Delta t\right)$, where $o\left(\Delta t\right)$ denotes second-order terms in Δt (i.e., $\lim_{\Delta t\to 0} o\left(\Delta t\right)/\Delta t = 0$). Therefore, aggregating across all intermediates, we have

$$Q\left(t+\Delta t\right) = (1+\lambda)\,Q\left(t\right)\eta\Delta t + \gamma Q\left(t\right)\alpha\Delta t + Q\left(t\right)\left(1-\alpha\Delta t-\eta\Delta t\right) + o\left(\Delta t\right).$$

Subtracting $Q\left(t\right)$, dividing by Δt and taking the limit as $\Delta t \to 0$, we obtain

$$g\left(t\right) = \frac{\dot{Q}\left(t\right)}{Q\left(t\right)} = g^* \equiv \lambda\eta - \alpha\left(1-\gamma\right). \tag{16}$$

Therefore, this analysis establishes the following proposition (proof in the text).

[14] It is also straightforward to see that this conclusion continues to be valid even if a scientist who has invented a higher-quality substitute variety maintains his patent following an exogenous improvement in quality because of an innovation for the active product (in this case, the denominator of (14) would be $r+\alpha$). See also the discussion of uniqueness in the next subsection.

Proposition 1 *In the above-described environment, the allocation where all research is directed at the active varieties ($\omega^* = 0$) and all industries have a gap of N steps between active and substitute varieties ($\mu_N^* = 1$) is a stationary equilibrium. In this equilibrium, the economy grows at the rate g^* given by (16).*

Clearly, the growth rate of the economy is decreasing in α. A switch from the active to the substitute variety of an intermediate causes a large drop in the contribution of this intermediate to output. Aggregate output in this economy is not stochastic because there is a large number of intermediates. Instead, intermediates where the active varieties disappear (at the flow rate α) lose a fraction $1 - \gamma$ of their contribution to output. Equation (16) also shows that the equilibrium growth rate is increasing in γ: the lower is γ, the more steep is the output drop of an intermediate experiencing a switch from the active to the substitute variety.

I next provide sufficient conditions that guarantee uniqueness of this stationary equilibrium. I then discuss the reasoning for the modeling assumptions used so far and then turn to an analysis of the optimality of this equilibrium.

3.3 Uniqueness

Can there be stationary equilibria other the one with $\omega^* = 0$ characterized in Proposition 1? The answer is yes because of the following mechanism: further research directed at substitute varieties increases the average quality of these varieties and makes such research more profitable. However, this mechanism is typically not strong enough to generate multiple equilibria. Moreover, there is a countervailing force pushing towards uniqueness, which is that more research directed at substitute varieties reduces the lifespan of each variety, thus making patents on such varieties less profitable. In this subsection, I provide sufficient conditions for uniqueness.

When there is research directed at substitute varieties, i.e., $\omega^* > 0$, then we cannot simply use equation (13) to determine the value of a patent on a substitute product of quality q'. This is because there may now be substitute varieties that are $n = 0, 1, ..., N$ steps behind the corresponding active variety (not just N steps behind as in the previous subsection). In that case, the exact value of n will determine the rate at which this patent may become redundant because of advances in the quality of the active variety (because q_j reaches $(1 + \lambda)\gamma^{-1} q_{j'}$). Therefore, we need to explicitly compute the value of a substitute variety of quality q' when it is n steps behind the active variety. This is given as

$$r\tilde{V}_n(q') = \alpha\left(V(q') - \tilde{V}_n(q')\right) + p_u\left(\tilde{V}_{n+1}(q') - \tilde{V}_n(q')\right) - p_d\tilde{V}_n(q'), \tag{17}$$

where $p_u \equiv \eta(1 - \omega)$ and $p_d \equiv \zeta\eta\omega$ as before and $\tilde{V}_{N+1}(q') \equiv 0$. It can be verified that when $\omega = 0$, this equation gives (13) and (14) for $n = N$; recall that (14) applies when $\omega = 0$ and when all substitute varieties are N steps behind. More generally, equation (17) highlights that there are three sources of changes in value: (i) a switch to the active variety status (at the flow rate α giving new value $V(q')$ instead of $\tilde{V}_n(q')$); (ii) a further improvement in the quality of the active variety, so that the gap increases to $n+1$ steps (at the flow rate p_u giving new value $\tilde{V}_{n+1}(q')$ instead of $\tilde{V}_n(q')$); and (iii) an innovation directed at the substitute varieties replacing this product (at the flow rate p_d giving value zero).

Using the fact that $\tilde{V}_{N+1}(q') \equiv 0$ and substituting for $p_u \equiv \eta(1-\omega)$ and $p_d \equiv \zeta\eta\omega$, we can recursively solve (17) to obtain

$$\tilde{V}_n(q') = \frac{\alpha}{r+\alpha+\zeta\eta\omega} V(q') \left[1 - \left(\frac{\eta(1-\omega)}{r+\alpha+\eta(1-\omega)+\zeta\eta\omega}\right)^{N+1-n}\right] \tag{18}$$

for $n = 1, ..., N$. Note that this is equivalent to

$$\tilde{V}_N(q') = \frac{\alpha}{r+\alpha+\eta(1-\omega)+\zeta\eta\omega} V(q').$$

For $n = 0$, the only difference is that there cannot be any further innovations in the substitute variety, thus

$$\tilde{V}_0(q') = \frac{\alpha}{r+\alpha+\zeta\eta\omega} V(q') \left[\frac{r+\alpha+\zeta\eta\omega}{r+\alpha+\eta(1-\omega)} + \eta(1-\omega)\left(1 - \left(\frac{\eta(1-\omega)}{r+\alpha+\eta(1-\omega)+\zeta\eta\omega}\right)^{N}\right)\right].$$

In a stationary allocation where a fraction ω of scientists are directing their research towards substitute varieties, the rate of replacement of active varieties will be $\eta(1-\omega)$, and thus (6) and (7) also need to be modified. In particular, with a similar reasoning to that in the previous subsection, these take the form

$$V(q) = \frac{\lambda q}{r+\alpha+\eta(1-\omega)}, \tag{19}$$

and

$$R^A(Q) = \eta \int_0^1 V(q(\nu,t))\, d\nu = \frac{\eta\lambda Q}{r+\alpha+\eta(1-\omega)}. \tag{20}$$

Next, turning to the expected return to a scientist directing his research to substitute varieties, we have

$$\begin{aligned} R^S &= \zeta\eta \sum_{n=1}^{N} \mu_n^* \tilde{V}_n(q') \\ &= \zeta\eta \sum_{n=1}^{N} \mu_n^* \tilde{V}_n\left((1+\lambda)^{-n} q\right), \end{aligned}$$

where in the first line the summation starts from $n = 1$, since there is no possibility of successful innovation for the fraction μ_0^* of intermediates where the gap is $n = 0$. The second line expresses the values as a function of the quality of the active variety, using the identity that if there is n step gap between active and substitute varieties, then $q' = (1+\lambda)^{-n} q$. Now using (12), (18) and (19), we can write

$$\begin{aligned} R^S(Q) &= \zeta\eta \frac{\lambda Q}{r+\alpha+\eta(1-\omega)} \frac{\alpha}{r+\alpha+\zeta\eta\omega} \Phi(\omega) \\ &= \zeta \frac{\alpha}{r+\alpha+\zeta\eta\omega} \Phi(\omega) R^A(Q), \end{aligned}$$

where again $Q \equiv \int_0^1 q_j(\nu)\, d\nu$ and the second line uses (20). In this expression,

$$\Phi(\omega) \equiv \sum_{n=1}^{N} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n} \left(\sum_{j=0}^{N} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-j}\right)^{-1} (1+\lambda)^{-n} \left[1 - \left(\frac{\eta(1-\omega)}{r+\alpha+\eta(1-\omega)+\zeta\eta\omega}\right)^{N+1-n}\right] \tag{21}$$

gives the expected quality of a substitute intermediate on which a researcher will build his innovation. It can be verified that $\Phi(0) = \gamma$ and $\Phi(\omega)$ is always strictly less than $1/(1+\lambda)$ (since the summation starts from $n = 1$). This implies that a sufficient condition for research directed at substitute not to be profitable is

$$\zeta\alpha \leq (1+\lambda)(r+\alpha). \tag{22}$$

This establishes (proof in the text):

Proposition 2 *Suppose that (22) holds. Then all research being directed at the active varieties ($\omega^* = 0$) and all intermediates having a gap of N steps between active and substitute varieties ($\mu_N^* = 1$) is the unique stationary equilibrium.*

3.4 Discussion of Modeling Assumptions

The framework in this section is designed as a "minimalist" dynamic model for the analysis of diversity of research. It clarifies the main modeling issues and attempts to communicate the main ideas of this note in a transparent manner. The Appendix presents a more standard model of endogenous technological change, which leads to results similar to those presented in this and the next sections. A natural question is whether an even simpler model could have been used to highlight the key economic mechanisms. I now briefly argue why this is not possible. In particular, there are five features of the model that are important either for the results or for simplifying the exposition: (1) the quality ladder structure; (2) a continuum of intermediates; (3) continuous time; (4) the feature that research cannot be directed to specific individual intermediates; (5) the characterization of the distribution of quality gaps between active and substitute varieties. I now explain why each of these is either necessary or greatly simplifies the analysis.

First, the quality ladder structure, for example, as in Aghion and Howitt (1992) or Grossman and Helpman (1991), is necessary for the results. This will become particularly clear in the next section, but the main idea can be discussed now. With the quality ladder structure, an innovation for the far future is not attractive because before the time to employ the innovation comes, another researcher is likely to have leapfrogged the product in question. In contrast, with a structure that incorporates horizontal innovations as in Romer (1990), following the invention of a new product (or machine), there are no further innovations replacing this product. This removes the externality that is central to the discussion here—the externality created on future versions of the same product or intermediate.

Second, the presence of a continuum of intermediates simplifies the analysis greatly by removing aggregate risk. Without this feature aggregate output would jump whenever there is an innovation or the active variety disappears. While in this case one could characterize the expected value of output, working with a continuum of intermediates simplifies the analysis both algebraically and conceptually.

Third, continuous time also simplifies the algebra. In particular, in discrete time, the relevant quantities become somewhat more involved because of the following two features: (a) the probability of success for an individual scientist depends on whether another scientist has been successful; in continuous time, the probability of two such events (success by this scientist and another) happening simultaneously vanishes, simplifying the expressions for expected returns from research; (b) the expression for the growth rate is also similarly simplified and takes the form given in (16), clearly highlighting the trade-off between research on active and substitute varieties.

Fourth, research is assumed to be directed to either active or substitute varieties, but not to specific intermediates. This is because, with the current formulation, profits are proportional to quality q and all researchers would prefer to direct their research to the variety with the highest q. The general model presented in the Appendix allows for a research technology that uses the final good (rather than the labor of scientists) and makes the cost of research proportional to the quality of the intermediate. With this formulation, the results do not depend on whether research can be directed to specific intermediates.

Finally, the most substantive aspect of the model is the characterization of the distribution of quality gaps between active and substitute varieties. While this introduces some amount of complication, it is necessary since the cost of lack of diversity is a large gap between the active and substitute varieties, which thus needs to be determined endogenously in equilibrium. An important modeling contribution of this note is to provide a tractable framework for an explicit characterization of the distribution of these gaps.

Several other features of the model are also adopted to simplify the exposition and will be relaxed in the Appendix. In particular, in the Appendix, I present an endogenous technological change model based on a quality ladder specification. The model in the Appendix does not assume linear preferences and perfect substitutions among different intermediates. In addition, the feature that innovations receive their full marginal contribution aggregate output when used in production is replaced with an explicit derivation of the profits of monopolistically competitive firms after they innovate. Finally, as also pointed out in the previous paragraph, this extended model further allows innovations to be directed not just to the active or the substitute varieties, but also to specific intermediates.

4 Optimal Technological Progress

In this section, I establish that when α is sufficiently large, the equilibrium is inefficient and the growth rate is inefficiently low. I then provide the economic intuition for this result.

4.1 Suboptimality of Equilibrium Technological Progress

Since all agents maximize the net present discounted value of output and are risk neutral (and idiosyncratic risks can be diversified), a natural measure of the optimality of the allocation of resources in this equilibrium is the expected value of output. Let us focus on this measure. First note that if $\alpha = 0$, we can ignore research on substitute varieties and the equilibrium allocation trivially coincides with the only feasible allocation. Thus the interesting case is when $\alpha > 0$. Suppose that a planner determines the allocation of scientists between research on the active and the substitute varieties. Consider the simple scenario where a fraction ω of the scientists are allocated to undertake research on substitute varieties.

The main result of this section is the following proposition.

Proposition 3 *Suppose that $\alpha > \alpha^*$, where*

$$\alpha^* \equiv \frac{\eta}{\zeta\gamma}. \tag{23}$$

Then the stationary equilibrium in Proposition 1 is suboptimal. In particular, starting with $\omega = 0$ a small increase in ω raises long-run output growth.

This proposition states that when potential switches from active to substitute technologies are sufficiently frequent, then some amount of "diversity in research," that is, research directed at both the active and the substitute varieties, is necessary to maximize steady-state equilibrium growth. Before presenting the proof of this proposition, note that it refers to "long-run" growth because it compares the stationary equilibrium growth rates to growth in an alternative stationary allocation. The Appendix provides a comparison of the net present discounted value of output taking into account the adjustment dynamics. It shows that the same conclusion as in Proposition 3 holds provided that $\alpha > \alpha^{**} > \alpha^{*}$.[15] The analysis in the Appendix gives the value of α^{**} as

$$\alpha^{**} \equiv \frac{r+\eta}{\zeta\gamma}. \tag{24}$$

As expected, when $r \to 0$, $\alpha^{**} \to \alpha^{*}$, since without discounting, the objectives of maximizing the long-run growth rate and the net present discounted value of output coincide.

To prove Proposition 3, let us first compute the relative quality gap between active and substitute varieties in a stationary allocation where a fraction ω of scientists are directing their research to substitute varieties. Let us again use Q to denote the average quality of active varieties (as defined in (3)) and let the average quality of the substitute varieties be ΓQ. Then in a stationary distribution given by $\langle \mu_0^*(\omega), \mu_1^*(\omega), ..., \mu_N^*(\omega)\rangle$, this gap parameter Γ as a function of ω can be written as

$$\begin{aligned}\Gamma(\omega) &= \sum_{n=0}^{N} (1+\lambda)^{-n} \mu_n^*(\omega) \\ &= \frac{\sum_{n=0}^{N} (1+\lambda)^{-n} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n}}{\sum_{n=0}^{N} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n}}. \end{aligned} \tag{25}$$

The first line of this expression defines $\Gamma(\omega)$ as the average relative quality of substitute varieties (relative to the active varieties), simply using the fact that this is given by a weighted average of the relative qualities of the substitute varieties of intermediates where substitutes have $n = 0, 1, ..., N$ step gaps and the weights are given by the stationary distribution fractions of intermediates with $n = 0, 1, ..., N$ gaps $(\mu_0^*(\omega), \mu_1^*(\omega), ..., \mu_N^*(\omega))$. The second line substitutes for $\mu_n^*(\omega)$ from (12).

It can be verified that $\lim_{\omega\to 0} \Gamma(\omega) = \gamma \equiv (1+\lambda)^{-N}$, consistent with the derivations in the previous section. Moreover, it can also be verified that $\Gamma(\omega)$ is continuously differentiable for all $\omega \in [0, 1)$, and straightforward differentiation gives its derivative as

$$\begin{aligned}\Gamma'(\omega) &= \frac{\left(\frac{1}{1-\omega}\right)^2 \zeta \sum_{n=1}^{N-1} (N-n)(1+\lambda)^{-n} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n-1}}{\sum_{n=0}^{N} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n}} \\ &\quad - \frac{\left(\frac{1}{1-\omega}\right)^2 \zeta \left(\sum_{n=1}^{N} (1+\lambda)^{-n} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n}\right) \sum_{n=1}^{N-1} (N-n) \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n-1}}{\left(\sum_{n=0}^{N} \left(\frac{\zeta\omega}{1-\omega}\right)^{N-n}\right)^2}.\end{aligned}$$

[15] It is also clear that $\alpha > \alpha^*$ is possible while $g^* = \lambda\eta - \alpha(1-\gamma) > 0$. For example, $\alpha = 4/3$, $\eta = 1$, $\lambda = 1.1$, $\gamma = 1/4$ and $\zeta = 4$ imply that $\alpha > \alpha^* = 1$, while $g^* = 0.1$, so that positive growth in the economy does not imply optimality.

And thus

$$\Gamma'(0) = \zeta\lambda (1+\lambda)^{-N} > 0.$$

With an identical argument to that in the previous section, the long-run (stationary allocation) growth rate of the economy is

$$g(\omega) = \lambda\eta (1-\omega) - \alpha (1 - \Gamma(\omega)),$$

with the only difference from (16) being that the first term is multiplied by $(1-\omega)$, reflecting the fact that not all scientists are working on active varieties, and the relative gap between active and substitute varieties is now $\Gamma(\omega)$ rather than γ. Therefore,

$$\begin{aligned} g'(0) &= -\lambda\eta + \alpha\Gamma'(0) \\ &= \alpha\zeta\lambda (1+\lambda)^{-N} - \lambda\eta. \end{aligned}$$

The result that whenever

$$\alpha > \alpha^* \equiv \frac{\eta (1+\lambda)^N}{\zeta} \equiv \frac{\eta}{\zeta\gamma},$$

equilibrium growth is too slow then follows immediately and establishes Proposition 3.

The Appendix shows that a similar result applies when we look at the adjustment of the distribution of gaps between active and substitute varieties following an increase in ω starting from $\omega = 0$.

4.2 Why Is the Equilibrium Suboptimal?

The externality that is not internalized in the equilibrium is the following: when a researcher undertakes an innovation either for an active or a substitute variety, it not only increases current output but also contributes to future output growth because it ensures that future innovations for this product will start from a higher base—each innovation increases existing quality by a proportional amount. However, the researcher does not capture these gains after its patent expires due to exogenous or endogenous technological change. This implies that every innovation creates positive externalities on all future innovators of the same (variety of the same) intermediate. When $\alpha = 0$, this externality does not affect the allocation of resources, since there is no choice concerning the direction of technological change and each scientist is already fully utilizing all of his capacity. However, when there is a choice between active and substitute varieties, this externality affects the relative private gains. In particular, the externality has a disproportionate effect on research directed at substitute varieties because this type of research is socially beneficial not for the immediate gains it generates but because it increases the quality of the substitute variety and creates a better platform for yet further innovation after the active variety disappears. Consequently, incentives to undertake research on such varieties are too low and there is not enough "diversity" in research.

This discussion also clarifies that the suboptimality identified here is a consequence of the patent system assumed in the analysis. This patent system is a stylized representation of the system of intellectual property rights used in most advanced economies, where a new product (process or technology) does not need to pay royalties to the previous innovations, provided that it improves existing technological know-how beyond a minimal required inventive step (or it improves over technologies that are more than 20 years old and are thus no longer patented).[16] Although, in practice, some innovations will need to make payments to previous patent holders, this does not change the thrust of the argument in this note; patent duration is capped at 20 years, and it is straightforward to extend the qualitative results presented here to a model with such limited patent payments.

In the context of the simple economy here, there exists an alternative patent system that can internalize the knowledge externalities and would prevent the inefficiency identified here. However, as discussed in the Introduction, this alternative patent system is both different from actual patent systems and is difficult to implement. Let us first discuss what this alternative patent system would have to look like. Since the externality is on future innovators, the patent system would have to involve a payment (e.g., royalty) from all future innovators in a particular line to the current innovator. For example, all innovations in laser technology or solid-state physics in the 20th century would have to pay royalties to Heisenberg, Einstein or Bohr. In practice, patent systems do not have this feature and once a new product or procedure is deemed to pass the originality (required step) requirement, it does not have to pay royalties to the innovators of the previous leading-edge technology, let alone to all innovations that invented the technologies that preceded the previous one.

5 Diversity and Technological Progress

In this section, I discuss how the diversity in the preferences, competences or beliefs of scientists affects equilibrium growth. I start with a simple variation on the model presented so far where scientists have a comparative advantage for active or substitute research. I then discuss another variation with heterogeneous beliefs.

5.1 Comparative Advantage

Suppose that each scientist has access to the same technology for innovating on active varieties (flow rate η), but in addition, if scientist i undertakes research on substitute varieties, then the

[16] See, for example, Scotchmer (2005).

flow rate at which he will succeed is given by $\varepsilon_i \eta$, where ε_i has a distribution across scientists given by $G(\varepsilon)$.[17] The variable ε_i captures *researcher diversity*—the diversity in the abilities, interests or beliefs of scientists concerning which research lines are likely to be successful in the future. Let us refer to ε_i as the "type" of the scientist. The model studied so far is a special case where $G(\varepsilon)$ has all of its mass at $\varepsilon = \zeta$. To ensure compatibility with the analysis in the previous section, let us assume that

$$\int_0^\infty \varepsilon dG(\varepsilon) = \zeta.$$

Let us denote the support of G as $\left[\zeta - \xi', \zeta + \xi\right]$ (with ξ', $\xi > 0$). We can think of $G(\varepsilon)$ as "highly" concentrated around ζ if ξ' and ξ are small. Let us define the notion of *greater diversity* (of scientists or researchers) as a mean-preserving spread of G involving an increase in ξ' and ξ.

Again consider a candidate equilibrium where all scientists direct their research towards active varieties. With an identical argument to that in Section 3, the value of undertaking research on active varieties for any scientist is given by (7) (since all scientists have the same productivity in research on active varieties). Similarly, the analysis leading up to (15) implies that the value of undertaking research towards substitute varieties for a researcher of type ε_i is

$$R^S(Q \mid \varepsilon_i) = \frac{\alpha \varepsilon_i \gamma}{r + \alpha + \eta} R^A(Q).$$

If G is highly concentrated around ζ, then research directed at substitute varieties will be unprofitable for all types. In particular, if

$$\xi \leq \xi^* \equiv \frac{r + \alpha + \eta}{\alpha \gamma} - \zeta, \tag{26}$$

then the allocation in which no scientist undertakes research directed towards substitute varieties is once again a stationary equilibrium.

Next, consider an increase in diversity, corresponding to a mean-preserving spread of G (in particular, an increase in ξ). For a sufficiently large change in G of this form, it will become profitable for some of the researchers with high ε's to start directing their research towards substitute varieties. When this happens, the form of the equilibrium resembles that discussed in subsection 3.3. In particular, there will clearly exist a threshold level $\bar{\varepsilon}$ such that scientists with type greater than $\bar{\varepsilon}$ will undertake research on substitute varieties, and thus the fraction of researchers working on active varieties will be $G(\bar{\varepsilon})$. The values of undertaking research towards substitute and active varieties in this case follow from the analysis in subsection 3.3. In particular, the value of undertaking research towards active varieties, when average quality of

[17] In other words, instead of a uniform innovation rate of $\zeta\eta$ for substitute varieties as in the previous two sections, now researcher i has innovation rate of $\varepsilon_i \eta$ if he directs his research to substitute varieties.

such varieties is Q, becomes

$$R^A(Q \mid \bar{\varepsilon}) = \eta \frac{\lambda Q}{r+\alpha+\eta G(\bar{\varepsilon})}. \tag{27}$$

The value of research directed towards substitute varieties is characterized as in subsection 3.3. In particular, let us define the equivalent of $\Phi(\omega)$ in (21) as

$$\Phi_\xi(\bar{\varepsilon}) \equiv \sum_{n=1}^{N} \left(\frac{\int_{\bar{\varepsilon}}^{\infty} \varepsilon dG(\varepsilon)}{G(\bar{\varepsilon})}\right)^{N-n} \left(\sum_{j=0}^{N} \left(\frac{\int_{\bar{\varepsilon}}^{\infty} \varepsilon dG(\varepsilon)}{G(\bar{\varepsilon})}\right)^{N-j}\right)^{-1} (1+\lambda)^{-n} \times \left[1 - \left(\frac{\eta G(\bar{\varepsilon})}{r+\alpha+\eta G(\bar{\varepsilon}) + \eta \int_{\bar{\varepsilon}}^{\infty} \varepsilon dG(\varepsilon)}\right)^{N+1-n}\right], \tag{28}$$

which takes into account that the probability of innovation in active and substitute varieties is no longer $\eta(1-\omega)$ and $\zeta\eta\omega$, but $\eta(1-G(\bar{\varepsilon}))$ and $\eta\int_{\bar{\varepsilon}}^{\infty} \varepsilon dG(\varepsilon)$. This function is also subscripted by ξ to emphasize its dependence on the distribution function G, particularly on its upper support, $\zeta+\xi$. Then, the value of undertaking research towards substitute varieties for a scientist of type ε_i (when the average quality of active varieties is Q) is

$$R^S(Q \mid \varepsilon_i, \bar{\varepsilon}) = \eta\varepsilon_i \frac{\alpha}{r+\alpha+\eta\int_{\bar{\varepsilon}}^{\infty} \varepsilon dG(\varepsilon)} \Phi_\xi(\bar{\varepsilon}) R^A(Q \mid \bar{\varepsilon}). \tag{29}$$

The equilibrium value of the threshold $\bar{\varepsilon}$ is then given by the solution to

$$R^S(Q \mid \varepsilon_i = \bar{\varepsilon}, \bar{\varepsilon}) = R^A(Q \mid \bar{\varepsilon}),$$

or by $\bar{\varepsilon}$ such that

$$\eta\bar{\varepsilon} \frac{\alpha}{r+\alpha+\eta\int_{\bar{\varepsilon}}^{\infty} \varepsilon dG(\varepsilon)} \Phi_\xi(\bar{\varepsilon}) = 1.$$

In general, such $\bar{\varepsilon}$ may not be unique. Nevertheless, it is clear that if ξ is greater than ξ^*, there does not exist a stationary equilibrium with no research directed at substitute varieties. Moreover, if ξ increases just above ξ^*, by the fact that $\Phi(\bar{\varepsilon})$ is continuous and $\Phi_\xi(\bar{\varepsilon}) = \gamma$ for $\xi \leq \xi^*$, the implications of this change will be identical to those of a small increase in ω starting from $\omega = 0$ in the baseline model. This argument thus establishes the following proposition (proof in the text).

Proposition 4 *In the above environment, consider a distribution of researcher diversity G_0 such that $\xi \leq \xi^*$ (as given by (26)). Then all research being directed at active varieties is a stationary equilibrium. Now consider a shift to G_1 with support $[\zeta - \xi_1', \zeta + \xi_1]$, where $\xi_1 > \xi^*$. This will increase the diversity of research in equilibrium and also raise the equilibrium growth rate provided that ξ_1 is sufficiently close to ξ^*.*

Proposition 4 shows that diversity of researchers will tend to increase the extent of diversity in research—that is, with more heterogeneous competences of researchers, equilibria will involve greater research effort being directed towards substitute varieties. Since the equilibrium of the baseline model studied in the previous two sections may have too much conformity and too little diversity, diversity from researchers may improve the rate of growth and technological progress in the economy. The proposition requires that ξ_1 is close to ξ^* for the equilibrium growth rate to increase. This is natural, since an extreme mean-preserving spread can induce (close to) half of all scientists to direct their research to substitute varieties, which will not necessarily increase growth.

5.2 Differences in Beliefs

A related but different interpretation of the analysis of the previous subsection and of Proposition 4 is also useful. Suppose that there is no difference in the abilities of the researchers and they all have a flow rate $\zeta\eta$ of undertaking successful innovations when their research is directed at substitute varieties. Instead, scientists have either different beliefs about the likelihood of switches between active and substitute varieties or obtain additional differential utility from undertaking research directed at targets different from the majority of other researchers. Suppose that ε again has a distribution given by G, with the same definition of ξ' and ξ, and let us also adopt the same definition of "greater diversity".

With this interpretation, the equations need to change a little, since a high ε researcher working on innovations in substitute varieties is not more productive. It is straightforward to repeat the same steps as above and conclude that as long as (26) holds, there will be no research directed at substitute varieties. However, when this condition does not hold, the equilibrium will take a slightly different form. In particular, (27) remains unchanged and gives the expected return to research directed at active varieties. Expected return to research on substitute varieties is, instead, given by

$$R^S\left(Q \mid \varepsilon_i, \bar{\varepsilon}\right) = \zeta\eta\varepsilon_i \frac{\alpha}{r+\alpha+\zeta\eta\left(1-G\left(\bar{\varepsilon}\right)\right)} \hat{\Phi}_\xi\left(\bar{\varepsilon}\right) R^A\left(Q \mid \bar{\varepsilon}\right),$$

with

$$\begin{aligned}\hat{\Phi}_\xi\left(\bar{\varepsilon}\right) \quad \equiv \quad & \sum_{n=1}^{N}\left(\frac{\zeta\left(1-G\left(\bar{\varepsilon}\right)\right)}{G\left(\bar{\varepsilon}\right)}\right)^{N-n}\left(\sum_{j=0}^{N}\left(\frac{\zeta\left(1-G\left(\bar{\varepsilon}\right)\right)}{G\left(\bar{\varepsilon}\right)}\right)^{N-j}\right)^{-1}(1+\lambda)^{-n} \times \\ & \left[1-\left(\frac{\eta G\left(\bar{\varepsilon}\right)}{r+\alpha+\eta G\left(\bar{\varepsilon}\right)+\zeta\eta\left(1-G\left(\bar{\varepsilon}\right)\right)}\right)^{N+1-n}\right].\end{aligned}$$

An increase in diversity again as similar effects. However, this slightly different interpretation also highlights an important point: the decisions of certain scientists to direct their research to substitute varieties may be "*non-profit maximizing*". This has two implications. First, it may be precisely the non-profit objectives of scientists that sometimes restore the diversity in research that may be socially beneficial and useful for more rapid technological progress. Second, with this interpretation, if researchers were employed in profit-maximizing organizations, there would be a conflict between the objectives of organizations (which would be to induce researchers to direct their efforts towards active varieties) and the wishes of the researchers themselves, and it would be the latter that is more useful for the society. This may then generate a justification for creating non-profit research centers (such as universities or independent research labs), where the diversity of researchers, rather than profit incentives, can guide the direction of their research effort.

6 Concluding Remarks

This note has presented a tractable dynamic framework for the analysis of the diversity of research. Using this framework, it is shown that equilibrium technological progress may feature too little diversity. In particular, it may fail to invest in "alternative" technologies, even if it is known that these technologies will become used at some point in the future. The economic intuition leading to this result is simple: innovations are made for current gain—the future benefits from these innovations are not fully internalized. This externality discourages research towards technologies that will bear fruit in the future because, in these research lines, current innovations are likely to be followed by further innovations before these technologies can be profitably marketed. A social planner wishing to maximize output (the net present discounted value of output or alternatively discounted utility) would choose a more diverse research portfolio and would induce a higher growth rate than the equilibrium allocation. I also showed how diversity of researchers—in particular, the presence of researchers with different interests, competences or ideas—can induce a more diverse research portfolio and thus increase economic growth.

The broader message is that the research process may, under certain circumstances, generate too much conformity and too little diversity—with all or the majority of scientists working to develop the same research lines. The model here emphasized one mechanism for such conformity: the greater profitability of developing currently-marketed products relative to technologies for the future. Other mechanisms may be equally important in practice. For example, learning from the success of others might create "herding," making the majority of the researchers follow early successes in a particular field. Or certain types of research may create greater externalities and more limited private returns, so that research becomes concentrated in "low-externality" fields. Depending on the exact mechanism leading to such lack of diversity in research, different types of policy and market remedies may be required. If the problem is one of lack of diversity, greater diversity of preferences, beliefs or competencies of researchers is likely to be socially useful. As discussed in the previous section, this might also suggest a justification for university-like organizations that encourage non-profit-seeking research behavior among scientists. More detailed theoretical and empirical investigations of whether and why there may be too much conformity or too little diversity in research and how the society might respond to this challenge are interesting areas for further study.

Appendix

Characterization of Optimal Policy

I now provide a characterization of the optimal policy, which involves comparing the entire path of output rather than the more straightforward comparisons of long-run growth rates reported in the text. With an argument identical to that in the text, the growth rate of average quality of output at any t (even when we are not in a stationary allocation) is

$$g(t) = \lambda\eta(1-\omega(t)) - \alpha(1-\Gamma(t)), \tag{30}$$

where

$$\Gamma(t) = \sum_{n=0}^{N} (1+\lambda)^{-n} \mu_n(t), \tag{31}$$

and $\mu_n(t)$ denotes the fraction of intermediates at time t with a gap of n steps between active and substitute varieties. This is similar to (25) and on the right-hand side we have the fractions of intermediates with different gaps (which are not necessarily the stationary equilibrium fractions). Correspondingly, with a slight abuse of notation, I use $\Gamma(t)$ rather than $\Gamma(\omega)$.

These fractions will evolve as a function of the time path of research devoted to substitute varieties, $[\omega(t)]_{t=0}^{\infty}$. In particular, with a reasoning identical to that leading to (8), the law of motion of these fractions is given by

$$\dot{\mu}_n(t) = \zeta\eta\omega(t)\mu_{n+1}(t) + \eta(1-\omega(t))\mu_{n-1}(t) - (\zeta\eta\omega(t) + \eta(1-\omega(t)))\mu_n(t)$$

for $n = 1, ..., N-1$, and in addition,

$$\dot{\mu}_N(t) = \eta(1-\omega(t))\mu_{N-1}(t) - \zeta\eta\omega(t)\mu_N(t)$$

and

$$\dot{\mu}_0(t) = \zeta\eta\omega(t)\mu_1(t) - \eta(1-\omega(t))\mu_0(t).$$

However, as noted in the text, one of these differential equations for μ is redundant, and in addition we have that

$$\sum_{n=0}^{N} \mu_n(t) = 1.$$

In what follows, it is most convenient to drop the differential equation for $\mu_{N-1}(t)$ and also write

$$\mu_{N-1}(t) = 1 - \sum_{n=0}^{N-2} \mu_n(t) - \mu_N(t). \tag{32}$$

Then, the differential equations that will form the constraints on the optimal control problem can be written as

$$\dot{\mu}_n(t) = \zeta\eta\omega(t)\mu_{n+1}(t) + \eta(1-\omega(t))\mu_{n-1}(t) - (\zeta\eta\omega(t) + \eta(1-\omega(t)))\mu_n(t) \tag{33}$$

for $n = 1, ..., N-3$,

$$\dot{\mu}_{N-2}(t) = \zeta\eta\omega(t)\left(1 - \sum_{n=0}^{N-2}\mu_n(t) - \mu_N(t)\right) + \eta(1-\omega(t))\mu_{N-3}(t) - (\zeta\eta\omega(t) + \eta(1-\omega(t)))\mu_{N-2}(t), \tag{34}$$

$$\dot{\mu}_N(t) = \eta(1-\omega(t))\left(1 - \sum_{n=0}^{N-2}\mu_n(t) - \mu_N(t)\right) - \zeta\eta\omega(t)\mu_N(t), \tag{35}$$

and

$$\dot{\mu}_0(t) = \zeta\eta\omega(t)\mu_1(t) - \eta(1-\omega(t))\mu_0(t). \tag{36}$$

Let us use boldface letters to denote sequences, i.e., $\boldsymbol{\omega} \equiv [\omega(t)]_{t=0}^{\infty}$. Therefore, the net present discounted value of output, taking into account adjustment dynamics, is given by

$$W(\boldsymbol{\omega}) = \int_{t=0}^{\infty} \exp(-rt)\, Q(t)\, dt.$$

The optimal policy will involve choosing $\boldsymbol{\omega}$ to maximize W subject to

$$\dot{Q}(t) = g(t)\, Q(t), \tag{37}$$

with $g(t)$ given by (30), and also subject to (32), (33)-(36) and (37). Without loss of any generality, let us normalize $Q(0) = 1$.

Given these differential equations, the optimal policy is determined as a solution to an optimal control problem with current value Hamiltonian, with appropriately defined costate variables and Lagrange multipliers. In particular, let the multipliers on (32) be $\chi(t)$, (33)-36) $\varphi_n(t)$ for $n = 0, 1, ..., N$, and on (37) $\kappa(t)$. Then

$$\begin{aligned} H(\boldsymbol{\omega}, \mathbf{Q}, \boldsymbol{\mu}, \boldsymbol{\varphi}, \boldsymbol{\kappa}, \boldsymbol{\chi}) = {} & \exp(-rt)\, Q(t) + \\ & +\kappa(t)\, g(t)\, Q(t) \\ & +\varphi_0[\zeta\eta\omega(t)\mu_1(t) - \eta(1-\omega(t))\mu_0(t)] \\ & +\sum_{n=1}^{N-3}\varphi_n(t)\left[\zeta\eta\omega(t)\mu_{n+1}(t) + \eta(1-\omega(t))\mu_{n-1}(t) - (\zeta\eta\omega(t) + \eta(1-\omega(t)))\mu_n(t)\right] \\ & +\varphi_{N-2}[\zeta\eta\omega(t)(1 - \sum_{n=0}^{N-2}\mu_n(t) - \mu_N(t)) \\ & \quad + \eta(1-\omega(t))\mu_{N-3}(t) - (\zeta\eta\omega(t) + \eta(1-\omega(t)))\mu_{N-2}(t))], \\ & +\varphi_N(t)[\eta(1-\omega(t))(1 - \sum_{n=0}^{N-2}\mu_n(t) - \mu_N(t)) - \zeta\eta\omega(t)\mu_N(t)] \\ & +\chi(t)[\sum_{n=0}^{N-2}\mu_n(t) + \mu_N(t) - 1]. \end{aligned}$$

Substituting from (31) and (32) into (30), we have

$$\begin{aligned} g(t) &= \lambda\eta(1-\omega(t)) \\ &\quad -\alpha\left[1-\sum_{n=0}^{N-2}(1+\lambda)^{-n}\mu_n(t)-(1+\lambda)^{-(N-1)}\left(1-\sum_{n=0}^{N-2}\mu_n(t)-\mu_N(t)\right)-(1+\lambda)^{-N}\mu_N(t)\right] \\ &= \lambda\eta(1-\omega(t))-\alpha\left(1-(1+\lambda)^{-(N-1)}\right) \\ &\quad +\sum_{n=0}^{N-2}\left((1+\lambda)^{-n}-(1+\lambda)^{-(N-1)}\right)\mu_n(t)-\alpha\lambda(1+\lambda)^{-N}\mu_N(t). \end{aligned}$$

Let us now write the necessary conditions for a continuous solution to this optimal control problem. We use $\mu_{N-1}(t) = 1-\sum_{n=0}^{N-2}\mu_n(t)-\mu_N(t)$ to simplify expressions.

For $\omega(t)$, we have

$$\begin{aligned} -&\quad \lambda\eta\kappa(t)Q(t) \\ +&\quad \varphi_0(t)\left[\zeta\eta\mu_1(t)+\eta\mu_0(t)\right] \\ +&\quad \sum_{n=1}^{N-1}\varphi_n(t)\left[\zeta\eta\left(\mu_{n+1}(t)-\mu_n(t)\right)+\eta\left(\mu_n(t)-\mu_{n-1}(t)\right)\right] \\ -&\quad \varphi_N(t)\left[\zeta\eta\mu_N(t)+\eta\mu_{N-1}(t)\right] \leq 0, \end{aligned} \tag{38}$$

where this condition is written as an inequality to allow for the solution to be at $\omega(t) = 0$ (and incorporating the fact that $\omega(t)$ will always be less than 1).

For $Q(t)$, we have

$$-\dot{\kappa}(t) = \exp(-rt)+\kappa(t)g(t), \tag{39}$$

For $\mu_0(t)$, we have

$$\begin{aligned} -\dot{\varphi}_0(t) &= \alpha\kappa(t)\left[1-(1+\lambda)^{-(N-1)}\right] \\ &\quad -\varphi_0(t)\eta(1-\omega(t))+\varphi_1(t)\zeta\eta\omega(t) \\ &\quad -\varphi_{N-2}(t)\zeta\eta\omega(t)-\varphi_N(t)\eta(1-\omega(t))+\chi(t). \end{aligned} \tag{40}$$

For $\mu_n(t)$ $(n = 1, ...N-2)$, we have

$$\begin{aligned} -\dot{\varphi}_n(t) &= \alpha\kappa(t)\left((1+\lambda)^{-n}-(1+\lambda)^{-(N-1)}\right) \\ &\quad -\varphi_n(t)\left[\zeta\eta\omega(t)+\eta(1-\omega(t))\right] \\ &\quad +\varphi_{n+1}(t)\eta(1-\omega(t))+\varphi_{n-1}(t)\zeta\eta\omega(t) \\ &\quad -\varphi_{N-2}(t)\zeta\eta\omega(t)-\varphi_N(t)\eta(1-\omega(t))+\chi(t). \end{aligned} \tag{41}$$

For $\mu_N(t)$, we have

$$\begin{aligned} -\dot{\varphi}_N(t) &= -\alpha\kappa(t)\lambda(1+\lambda)^{-N} \\ &\quad -\varphi_{N-2}(t)\zeta\eta\omega(t) \\ &\quad +\varphi_N(t)\left[\eta(1-\omega(t))-\zeta\eta\omega(t)\right]+\chi(t). \end{aligned} \tag{42}$$

In addition, we have a set of transversality conditions corresponding to each of the state variables.

Now suppose that we start at $t=0$ with $\mu_n(0)=0$ for all $n=0,1,...,N-1$, and thus naturally, $\mu_N(0)=1$. We will now characterize the conditions under which $\omega(t)=0$ for all t is not optimal. Suppose, to obtain a contradiction, that starting from such an allocation $\omega(t)=0$ for all t is optimal. Let us also define (as in the text)

$$g^* \equiv \lambda\eta-\alpha(1-\gamma).$$

Since $\mu_n(0)=0$ for all $n=0,1,...,N-1$, (32) is slack, thus $\chi(t)=0$. Moreover, since $\mu_n(t)=0$ for all t and $n=0,1,...,N-2$, we can ignore the evolution of $\varphi_n(t)$ (for $n=0,1,...,N-2$). Thus we can simply focus on the evolution of the two costate variables, $\kappa(t)$ and $\varphi_N(t)$. Since $\omega(t)=0$ for all t and $\mu_n(t)=0$ for all t and $n=0,1,...,N-2$, their evolution is given by the following two differential equations:

$$-\dot{\kappa}(t)=\exp(-rt)+g^*\kappa(t), \tag{43}$$

and (using also the fact that $\gamma\equiv(1+\lambda)^{-N}$)

$$\dot{\varphi}_N(t)=\alpha\lambda\gamma\kappa(t)+\eta\varphi_N(t). \tag{44}$$

Since (43) only depends on $\kappa(t)$, it has a unique solution of the form

$$\kappa(t)=c_K\exp(-g^*t)+\frac{\exp(-rt)}{r-g^*},$$

where c_K is a constant of integration. The transversality condition corresponding to $Q(t)$ requires that $r>g^*$ (which we assume) and that $c_K=0$, thus

$$\kappa(t)=\frac{\exp(-rt)}{r-g^*}. \tag{45}$$

Now using (45), the second differential equation (44) also has a unique solution

$$\varphi_N(t)=c_N\exp(\eta t)-r\alpha\lambda\gamma\frac{\exp(-rt)}{(r-g^*)(r+\eta)},$$

where c_N is a constant of integration, again set equal to 0 by the transversality condition. Therefore,

$$\varphi_N(t)=-\alpha\gamma\lambda\frac{\exp(-rt)}{(r-g^*)(r+\eta)}. \tag{46}$$

Combining (45) and (46) with (38) and recalling the normalization that $Q(0) = 1$, we have that a necessary condition for $\omega(t) = 0$ for all t to be an optimal solution is

$$-\lambda\eta\frac{\exp(-rt)}{r-g^*}\exp(g^*t) + \zeta\eta\alpha\lambda\gamma\frac{\exp(-rt)}{(r-g^*)(r+\eta)} \leq 0$$

for all t. Now let us look at this condition when $t = 0$, which is equivalent to

$$\zeta\gamma\alpha \leq (r+\eta).$$

Therefore, if

$$\alpha > \alpha^{**} \equiv \frac{r+\eta}{\zeta\gamma},$$

the candidate solution is not optimal and we conclude that the policy that maximizes the discounted value of income (or utility) will involve directing some research towards substitute varieties, proving the claim in the text.

General Model

I now present a more general environment building on Aghion and Howitt (1992), Grossman and Helpman (1991), and the textbook endogenous technological change model presented in Acemoglu (2009). This model generalizes the baseline environment presented in the text and shows that several assumptions used in the text are unnecessary for the results. The environment is again in continuous time and aggregate output is produced by combining a continuum of intermediates. As in the text, each intermediate ν comes in a countably infinite number of varieties, denoted by $j_1(\nu)$, $j_2(\nu)$,...., again one of those being active at any point in time. Let us focus on the active variety $j(\nu)$, and the next-in-line (substitute) variety $j'(\nu)$. Qualities are again denoted by $q_j(\nu,t) > 0$ and $q_{j'}(\nu,t) > 0$, and evolve endogenously. The production function for aggregate output at time t is

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^1 q_j(\nu,t)x_j(\nu,t|q)^{1-\beta}d\nu\right)L^\beta,$$

where $x_j(\nu,t|q)$ is the quantity of the active variety of intermediate ν (of quality $q_j(\nu,t)$, so that $x_j(\nu,t|q)$ is short for $x_j(\nu,t|q_j(\nu,t))$) purchased at time t and L is total labor, supplied inelastically. This production function exhibits constant returns to scale to intermediates and labor. As in the main text, there is a quality ladder for each intermediate (of active and substitute varieties), equi-distant rungs. Thus each innovation takes the machine quality up by one rung on this ladder, so that following each improvement quality increases by a proportional amount $1+\lambda > 1$. Also as in the main text, we have that if $q_{j'}(\nu,t) = \gamma q_j(\nu,t)$ and $q_j(\nu,t)$ increases to $q_j(\nu,t+) = (1+\lambda)q_j(\nu,t)$, then the quality of the substitute variety also increases to $q_{j'}(\nu, t+) = \gamma(1+\lambda) q_j(\nu, t)$. Similarly, we also continue to assume that the quality of the next-in-line substitute variety can be no less than $\gamma q_j(\nu, t)$.

New machine vintages are again invented by R&D. R&D effort can be directed to any of the different intermediates and to active or substitute varieties. Here, let us suppose that R&D uses the final good as input (rather than scientists). In particular, if $Z(\nu, t)$ units of the final good are spent for research to create an intermediate of quality $q(\nu, t)$, then it generates a flow rate

$$\frac{\eta Z(\nu, t)}{q(\nu, t)}$$

of innovation. This specification implies that one unit of R&D spending is proportionately less effective when applied to a more advanced intermediate, which ensures that research will be directed to lower quality as well as higher quality intermediates.

Suppose also that there is free entry into research, thus any firm or individual can undertake research on any of the varieties of any of the intermediates.

Once a particular machine of quality $q(\nu, t)$ has been invented, any quantity of this machine can be produced at marginal cost $\psi q(\nu, t)$. The assumption that the marginal cost is proportional to the quality of the machine is natural, because producing higher-quality machines should be more expensive. I normalize $\psi \equiv 1 - \beta$ without any loss of generality.

Let us also suppose that the consumer side of this economy admits a representative household with the standard CRRA preferences, in particular, at time $t = 0$ maximizing

$$\int_0^\infty \exp(-rt) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt.$$

Finally, the resource constraint of the economy is

$$X(t) + Z(t) + C(t) \leq Y(t),$$

where $X(t) \equiv \int_0^1 x_j(\nu, t|q) \, d\nu$ is the total amount of the final good spent on the production of the intermediate. Thus, this constraint requires that the amounts devoted to intermediate production, R&D and consumption should not exceed total output.

Household maximization implies the familiar Euler equation,

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho). \tag{47}$$

A firm that has access to the highest quality active variety of intermediate will be the monopoly supplier of intermediate and will make profits, denoted by $\pi(\nu, t|q)$ for intermediate $\nu \in [0, 1]$ of quality q. The value of this firm is given by a Hamilton-Jacobi-Bellman equation similar to (5), in particular, taking into account possible changes in the value functions over time and denoting the endogenously-determined interest rate at time t by $r(t)$, this is

$$r(t) V_j(\nu, t|q) - \dot{V}_j(\nu, t|q) = \pi(\nu, t|q) - (\alpha + z(\nu, t|q)) V_j(\nu, t|q),$$

which again takes into account the destruction of this value due to both further innovations (at the flow rate $z(\nu, t|q)$) and switches away from the active variety (at the flow rate α).

Factor markets are assumed to be competitive.

Let us start with the aggregate production function for the final good producers. Straightforward maximization gives the demand for each intermediate as follows:

$$x(\nu, t|q) = \left(\frac{q(\nu, t)}{p^x(\nu, t|q)}\right)^{1/\beta} L \text{ for all } \nu \in [0, 1] \text{ and } t,$$

where $p^x(\nu, t|q)$ refers to the price of machine of variety ν of quality q at time t. This is an iso-elastic demand curve and the monopoly producers of the highest quality intermediate (of the active variety) will wish to set monopoly price that is a constant markup over marginal cost. However, we also need to ensure that this monopoly price is not so high as to make the next best vintage profitable. The following assumption is enough to ensure this:

$$\lambda \geq \left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}} - 1. \tag{48}$$

This then guarantees that the profit-maximizing price is

$$p^x(\nu, t|q) = q(\nu, t), \tag{49}$$

and thus the equilibrium involves

$$x(\nu, t|q) = L. \tag{50}$$

Consequently, the flow profits of the firm selling intermediate of quality $q(\nu, t)$ is

$$\pi(\nu, t|q) = \beta q(\nu, t) L. \tag{51}$$

Using this expression, total output in the economy is

$$Y(t) = \frac{1}{1-\beta} Q(t) L, \tag{52}$$

where, with the same convention that j refers to the active variety,

$$Q(t) \equiv \int_0^1 q_j(\nu, t) d\nu \tag{53}$$

is the average total quality of machines. This analysis thus shows that the relevant expressions here, in particular, the form of the derived production function, (52), and the returns from having access to the highest quality, (53), are very similar to those in the text, but are derived from the aggregation of profit-maximizing micro behavior. It is also important that, as in the text, the $q(\nu, t)$'s are stochastic, but their average $Q(t)$ is deterministic with a law of large numbers type of reasoning (since the realizations of the quality of different machine lines are independent). Total spending on intermediates can also be computed as

$$X(t) = (1-\beta) Q(t) L. \tag{54}$$

Finally, the equilibrium wage rate, given by the marginal product of labor, is

$$w(t) = \frac{\beta}{1-\beta} Q(t). \tag{55}$$

The free-entry condition for active varieties, written in complementary slackness form, is

$$\eta V_j(\nu, t|q) \leq q \text{ and } \eta V_j(\nu, t|q) = q \text{ if } Z(\nu, t) > 0. \tag{56}$$

Next, we can also write the value function for substitute varieties. To do this, let us again focus on equilibrium in which there is zero R&D toward substitute varieties. In that case, with the reasoning similar to that in the main text, we have that the value of a substitute variety of quality q' is given by

$$r(t) V_{j'}(\nu, t|q') - \dot{V}_{j'}(\nu, t|q') = \alpha \left(V_j(\nu, t|q') - V_{j'}(\nu, t|q') \right) - z^* V_{j'}(\nu, t|q'),$$

where z^* is the equilibrium rate of innovation in active varieties. The relevant free entry condition in this case can then be written as

$$\eta V_{j'}(\nu, t|q') \leq q'.$$

First, note that in the candidate equilibrium, both the value functions of active and substitute varieties will be independent of time and also of ν, and can be written as $V(q)$ and $\tilde{V}(q')$. Then, an identical analysis to that in the text implies that for all $\alpha > 0$, the free entry condition for the substitute varieties of all intermediates will be slack. In this case, we have,

$$V(q) = \frac{\lambda \beta q}{r + \alpha + z}.$$

Free entry into research for active varieties requires

$$\eta V(q) \leq q,$$

or

$$\frac{\eta \beta}{r + \alpha + z} \leq 1.$$

Free entry into research for non-leading vintage can be expressed as

$$\begin{aligned} \tilde{V}(q) &\leq \eta q \\ \frac{\alpha}{r + \alpha + z} V(q) &\leq \eta q, \end{aligned}$$

which will always be satisfied as strict inequality whenever the free entry condition for active varieties is satisfied.

With an argument similar to that in the text, the growth rate of average quality of technology is

$$\frac{\dot{Q}(t)}{Q(t)} = \lambda z^* + \alpha(\gamma - 1).$$

Moreover, in this allocation, the consumer Euler equation implies

$$r = \rho + \theta g,$$

where g is the growth rate of output and consumption.

The free entry condition then can be written as

$$\frac{\eta\lambda}{\rho + \theta(\lambda - 1)z + \theta\alpha(\gamma - 1) + z + \alpha} = 1.$$

Thus:

$$z^* = \frac{\eta\lambda - \rho - \theta\alpha(\gamma - 1) - \alpha}{1 + \theta(\lambda - 1)}.$$

$$g^* = \frac{\eta\lambda - \rho - \theta\alpha(\gamma - 1) - \alpha}{\theta + \lambda^{-1}} - \alpha(1 - \gamma).$$

Then, the growth rate of output will be positive, i.e., $g^* > 0$, if

$$\eta\lambda - \rho + \theta\alpha(1 - \gamma) - \alpha > \theta\alpha(1 - \gamma) + \lambda^{-1}\alpha(1 - \gamma).$$

The rest of the analysis can be carried out in a manner similar to that in the text.

References

Acemoglu, Daron, *Introduction to Modern Economic Growth*, Princeton University Press, Princeton, New Jersey, 2009.

Acemoglu, Daron, Kostas Bimpikis and Asuman Ozdaglar, "Experimentation, Innovation and Patents" National Bureau of Economic Research, Working Paper No. 14408, 2008.

Aghion, Philippe, Mathias Dewatripont and Jeremy Stein, "Academic Freedom, Private Sector Focus and the Process of Innovation" Harvard University mimeo, 2007.

Aghion, Philippe and Peter Howitt, "A Model of Growth Through Creative Destruction" *Econometrica*, 110, (1992), 323-351.

Aghion, Philippe and Jean Tirole (1994) "On the Management of Innovation" *Quarterly Journal of Economics*, 109, (1994), 1185-1209.

Arrow, Kenneth J. "The Economic Implications of Learning by Doing." *Review of Economic Studies,* 29, (1962), 155-173.

Arthur, W. Brian "Competing Technologies, Increasing Returns and Lock-in by Historical Events." *Economic Journal*, 99, (1989), 116-131.

Bramoullé, Yann and Gilles Saint-Paul, "Research Cycles" Toulouse School of Economics, mimeo, 2008.

Bronfenbrenner, Martin (1966) "Trends, Cycles, and Fads in Economic Writing" *American Economic Review*, 56, (1966), 538-552.

Brock, William and Steven Durlauf, "A Formal Model of Theory Choice in Science" *Economic Theory*, 14, (1999), 113-130.

Crosby, Alfred, *The Children of the Sun: A History of Humanity's Unappeasable Appetite for Energy*, WW Norton, New York, 2006.

Grossman, Gene and Elhanan Helpman, *Innovation and Growth in the Global Economy,* Cambridge, MA, MIT Press, 1991.

International Energy Agency, *Key World Energy Statistics 2008*, http://www.iea.org/textbase/nppdf/free/2008/key_stats_2008.pdf

Jones, Benjamin F. "The Burden of Knowledge and the Death of the Renaissance Man: Is Innovation Getting Harder?" forthcoming *Review of Economic Studies*, 2009.

Katz, Michael L. and Carl Shapiro "Technology Adoption in the Presence of Network Externalities." *Journal of Political Economy*, 94, (1986), 822-841.

Murray, Fiona, Philippe Aghion, Mathias Dewatripont, Julian Kolev and Scott Scott (2008) "Of Mice and Academics: Examining the Effect of Openness on Innovation" MIT Sloan school mimeo, 2008.

Norris, J. R. *Markov Chains*, Cambridge University Press, Cambridge, UK, 1998.

Roberts, Paul, *End of Oil: On the Edge of a Perilous New World*, First Mariner Books, New York, 2005.

Romer, Paul M., "Endogenous Technological Change" *Journal of Political Economy*, 98, (1990), S71-S102.

Scotchmer, Suzanne, *Innovations and Incentives,* MIT Press, Cambridge, Massachusetts, 2005.

Stephan, Paula F. and Sharon G. Levin *Striking the Mother Lode in Science: the Importance of Age, Place, and Time,* Oxford University Press, New York, 1992.

Sunstein, Cass, "On Academic Fads and Fashions" *Michigan Law Review*, 99, (2001), 1251-1264.